# Some observation on developments in lossy TLM

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**Introduction**
If a subject is said to have reached maturity when it is no longer possible for an individual researcher to keep abreast of all, then surely TLM has already achieved that status. Thus, this paper covers only a small part of a burgeoning activity. We will start by outlining the topics which have kept us busy at UEA during the last year and will conclude with some observations on topics which may become relevant to TLM in the near future.

During the year grant income dried up. However, that did not completely halt progress. The collaboration with David Chen in Nova Scotia continued. Kaz Chichlowski went to work for BT, while Ciarán Kenny and Abhi Chakrabarti received their PhDs and are now based in Colorado and Ghent respectively. And of course, the extremely valuable contact with Xiang Gui in Calgary continues and he has made significant contributions to the work reported here.

**1. Matched boundaries**
This problem was first suggested by David Chen who has been working on the Berenger approach to perfectly matched load boundaries (PMLs) [1] and the work has been reported at the recent ACES conference [2]. It was reckoned that it should be possible to match a lossless and a lossy section of transmission line.

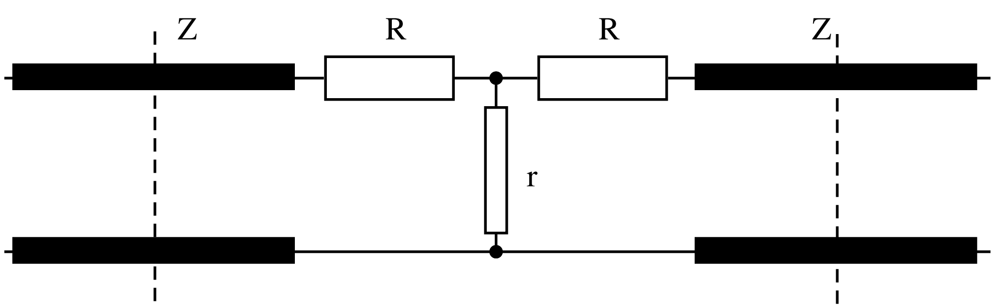


Figure 1 The matching of a lossy node with lossless nodes. So long as the node is symmetrical then it provides a match for pulse flow in both directions.

The condition for matching is:

$$\left[\frac{1}{r} + \frac{1}{R+Z}\right]^{-1} + R = Z \qquad (1)$$

This node will give rise to dispersion and for that reason we sought to have both matching and fulfilment of the Heaviside condition which states that a line of inductance, L, capacitance, C, resistance, 2R and conductance, G will be distortionless if:

$$\frac{L}{C} = \frac{2R}{G} \qquad (2)$$

For the node in figure 1 with a characteristic impedance, Z and conductance, G = 1/r this yields

$$Z^2 = 2R\,r \qquad (3)$$

With normalised characteristic impedance the conditions for distortionless matching (eqns (1) and (3)) give non-negative solutions

$$R = -r + \sqrt{r^2 + 1} \ \text{(matching)}$$
$$R = \frac{1}{2r} \ \text{(distortionless)} \qquad (4)$$

Sadly these two expressions for R do not coincide for any value of $R < \infty$ so that as it stands it is not possible to achieve distortionless matching.

This situation can be radically altered by the inclusion of an open circuit half-length (capacitive) stub transmission line which is placed at the node centre as shown in figure 2.

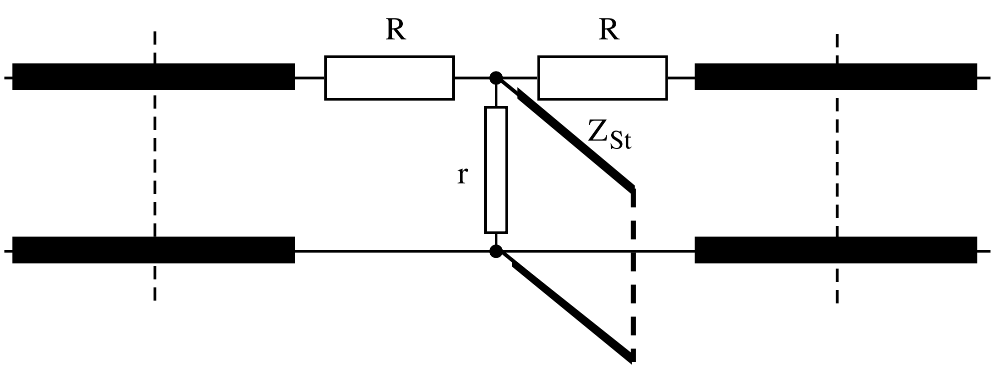


Figure 2 Lossy diffusion node with capacitive stub

The inclusion of the stub means that the total capacitance of the node is increased and the Heaviside condition (assuming normalised impedances) now becomes

$$\left[\frac{Z_{St}}{Z_{St}+1}\right]^2 = 2R\,r \tag{5}$$

The equivalent expression for the matching condition is

$$R = \frac{-r\,Z_{St} + \sqrt{r^2\,Z_{St}^2 + Z_{St}^2 + 2\,r\,Z_{St} + r^2}}{r + Z_{St}} \tag{6}$$

These equations are plotted in figure 3 for the case where $Z_{St} = 2$ and confirms that there is a point of coincidence which can be determined iteratively as r = 0.28432.

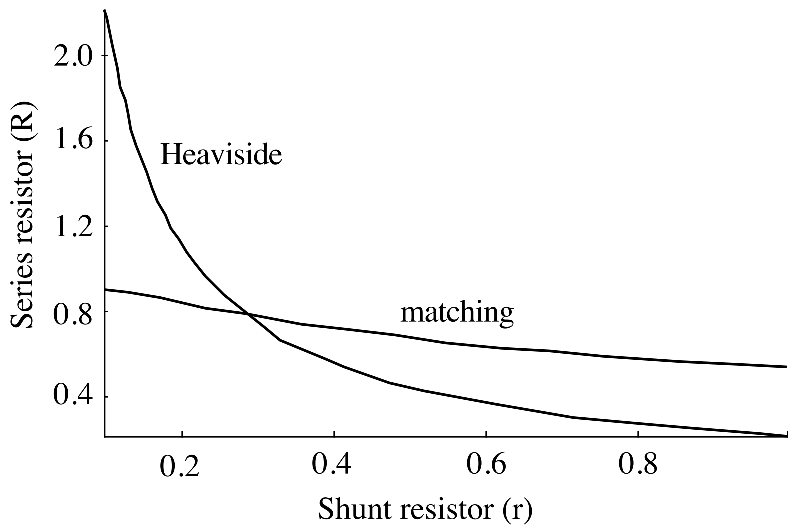


Figure 3 Plots of the matching and distortionless dependence of R as a function of r for a node with a capacitive stub $Z_{St} = 2$.

The incorporation of several of these nodes will effectively provide an artificial boundary and eliminate any spurious return signal. However, there is some evidence of band-limiting which has yet to be researched.

**2. A parametric analysis for consistency**

This section summarises the work of Ciarán Kenny and Richard Harvey which was presented at the recent ACES conference [3]. The nodal potential for a 2-dimensional formulation is given in very general terms:

$$_{k+1}\phi(x,y) = A \sum_{m=1}^{4} {}_{k+1}{}^{i}V_m(x,y) + B\, {}_{k+1}{}^{i}V_5(x,y) \tag{7}$$

where m= 1, 2, 3, 4 are the link-line directions; '5' refers to a stub (if present) and A and B are constants

Scatter and connect are expressed in terms of the potential at adjacent nodes:

$$
\begin{aligned}
{}_{k+1}{}^{i}V_1(x,y) &= {}_{k}{}^{s}V_3(x-1,y) = a\,{}_{k}\phi(x-1,y) + b\,{}_{k}{}^{i}V_3(x-1,y) \\
{}_{k+1}{}^{i}V_2(x,y) &= {}_{k}{}^{s}V_4(x,y+1) = a\,{}_{k}\phi(x,y+1) + b\,{}_{k}{}^{i}V_4(x,y+1) \\
{}_{k+1}{}^{i}V_3(x,y) &= {}_{k}{}^{s}V_1(x+1,y) = a\,{}_{k}\phi(x+1,y) + b\,{}_{k}{}^{i}V_1(x+1,y) \\
{}_{k+1}{}^{i}V_4(x,y) &= {}_{k}{}^{s}V_2(x,y-1) = a\,{}_{k}\phi(x,y-1) + b\,{}_{k}{}^{i}V_2(x,y-1) \\
{}_{k+1}{}^{i}V_5(x,y) &= {}_{k}{}^{s}V_5(x,y) = c\,{}_{k}\phi(x,y) + d\,{}_{k}{}^{i}V_5(x,y)
\end{aligned}
\tag{8}
$$

In this case a, b, c and d are constants.

These can be substituted into eqn (7) to give

$$
\begin{aligned}
{}_{k+1}\phi(x,y) = {} & Aa\left[{}_{k}\phi(x-1,y) + {}_{k}\phi(x,y+1) + {}_{k}\phi(x+1,y) + {}_{k}\phi(x,y-1)\right] \\
& + Ab\left[{}_{k}{}^{i}V_3(x-1,y) + {}_{k}{}^{i}V_4(x,y+1) + {}_{k}{}^{i}V_1(x+1,y) + {}_{k}{}^{i}V_2(x,y-1)\right] \\
& + Bc\,{}_{k}\phi(x,y) + Bd\,{}_{k}{}^{i}V_5(x,y)
\end{aligned}
\tag{9}
$$

Shifting in space and time allows us to remove all the incident pulses from this equation.

$$
\begin{aligned}
{}_{k}{}^{i}V_1(x,y) &= {}_{k-1}{}^{s}V_3(x,y) = a\,{}_{k-1}\phi(x,y) + b\,{}_{k-1}{}^{i}V_3(x,y) \\
{}_{k}{}^{i}V_2(x,y) &= {}_{k-1}{}^{s}V_4(x,y) = a\,{}_{k-1}\phi(x,y) + b\,{}_{k-1}{}^{i}V_4(x,y) \\
{}_{k}{}^{i}V_3(x,y) &= {}_{k-1}{}^{s}V_1(x,y) = a\,{}_{k-1}\phi(x,y) + b\,{}_{k-1}{}^{i}V_1(x,y) \\
{}_{k}{}^{i}V_4(x,y) &= {}_{k-1}{}^{s}V_2(x,y) = a\,{}_{k-1}\phi(x,y) + b\,{}_{k-1}{}^{i}V_2(x,y) \\
{}_{k}{}^{i}V_5(x,y) &= {}_{k-1}{}^{s}V_5(x,y) = c\,{}_{k-1}\phi(x,y) + d\,{}_{k-1}{}^{i}V_5(x,y)
\end{aligned}
\tag{10}
$$

These can be substituted back to give a two-step routine but there will still be incident components. These could in turn be replaced to give a three-step routine but again there will be incident terms. If we impose the condition that $b^2 = d^2$ then all the incident voltage terms cancel and we then obtain an expression in terms of the nodal voltages only:

$$ {}_{k+1}\phi(x,y) = Aa\left[ {}_{k}\phi(x-1,y) + {}_{k}\phi(x,y+1) + {}_{k}\phi(x+1,y) + {}_{k}\phi(x,y-1) \right] $$

$$ + \ (4Aab + Bcd + b^2)\ {}_{k-1}\phi(x,y) \qquad (11) $$

We can now take this expression and compare it term-by-term with conventional TLM to obtain values for the general constants A, B, a, b, c, and d

The condition $b^2 = d^2$ is equivalent to

$$ \frac{R\,Z}{R_s\,Z_s} = \frac{R^2 + Z^2}{R_s{}^2 + Z_s{}^2} \qquad (12) $$

Thus if R, Z and Zs are chosen then $R_s$ can only take the values $\frac{Z\,Z_s}{R}$ or $\frac{R\,Z_s}{Z}$

The expressions in eqn (9) can also be compared term-by-term with finite difference formulations and the requirements for consistency between both lossless and lossy TLM and their finite difference equivalents can be determined.

All of these attempts to develop comparisons between lossy TLM and finite difference formulations of the diffusion equation have so far ignored the question of initial conditions. Unless these are identical in the two approaches then there is no guarantee that they will converge to the same solution. An initial condition $f(x,y) = {}_{0}\phi(x,y)$ can be applied to a finite difference approximation of the diffusion equation which can then be stepped forward to give:

$$ {}_{1}\phi(x,y) = f(x,y) + r\left[ f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y) \right] \qquad (13) $$

Stepping forward with the generalised TLM scheme gives:

$$ {}_{1}\phi(x,y) = Aa\left[ f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) \right] $$

$$ + \ Ab\left[ {}_{k}{}^{i}V_3(x-1,y) + {}_{k}{}^{i}V_4(x,y+1) + {}_{k}{}^{i}V_1(x+1,y) + {}_{k}{}^{i}V_2(x,y-1) \right] $$

$$ + Bc\, f(x,y) + Bd\ {}_{k}{}^{i}V_5(x,y) \qquad (14) $$

The comparison for the diffusion equation forces us to have b = d = 0. This means that it is the nodal values and not the individual pulse values which satisfy the initial conditions. Thus a shunt TLM mesh can be excited by pulses on some or all lines with any weighting provided that they sum to ${}_0f(x,y)$. In the case of lossless TLM an equivalent treatment shows that it is essential to correctly define individual initial pulse values if the two approaches are to be consistent.

**3. Drift-diffusion**

The convection, advection or drift-diffusion equation in one dimension is given by:

$$\frac{\partial C(x,t)}{\partial t} = -\frac{\partial (v\, C(x,t))}{\partial x} + \frac{\partial}{\partial x}\left[D\frac{\partial C(x,t)}{\partial x}\right] \qquad (15)$$

which if the drift velocity, v and diffusion constant, D are invariant becomes:

$$\frac{\partial C(x,t)}{\partial t} = -v\frac{\partial (C(x,t))}{\partial x} + D\frac{\partial^2 C(x,t)}{\partial x^2} \qquad (16)$$

In the modified algorithm for drift and diffusion which was proposed by Saleh [4] the nodal voltage ${}_k\phi_m$ is defined in terms of incident pulses and current generator which depends on the mean difference between neighbouring nodes $\phi_{x+1}$ and $\phi_{x-1}$. Saleh used the adjacent values at the previous iteration interval.

$${}_k\phi(x) = ({}_kV^i{}_L(x) + {}_kV^i{}_R(x)) + I(x)\left[\frac{R+Z}{2}\right] \qquad (17)$$

where $I(x) = g_m\left[\frac{{}_{k-1}\phi(x+1) - {}_{k-1}\phi(x-1)}{2}\right]$ and $g_m Z = \frac{-v}{(\Delta x/\Delta t)}$

The stability of algorithms was investigated using the standard von Neumann stability analysis. This represents the initial condition as a series of Fourier modes and checks whether the solution remains bounded. An expression was developed for the amplification factor, $\xi$ and the conditions when $|\xi| \leq 1$ were then investigated. This can be summarised within the Courant-Friedrichs-Lewy stability criterion [5] which states that diffusion-convection algorithms are stable provided the following condition is satisfied:

$$1 \geq 2\frac{D\,\Delta t}{\Delta x^2} \geq g_m{}^2 \qquad (18)$$

More detailed information can be obtained by replacing the potentials in the multi-step algorithm by a wave of the form $e^{j\lambda m\Delta x}\xi^k$ (where $\lambda$ is the wavelength and k is the iteration number). When this is applied to the TLM algorithm for drift-diffusion and simplified we obtain the following equation:

$$\xi^3 - 2(\tau\cos\lambda\Delta x + jB\sin\lambda\Delta x)\xi^2 - (2\rho - 1)\xi + 2jB(2\rho - 1)^2 = 0 \qquad (19)$$

where $B = \dfrac{g_m(R + Z)}{4}$ or $\dfrac{g}{4\tau}$ if $Z = 1$

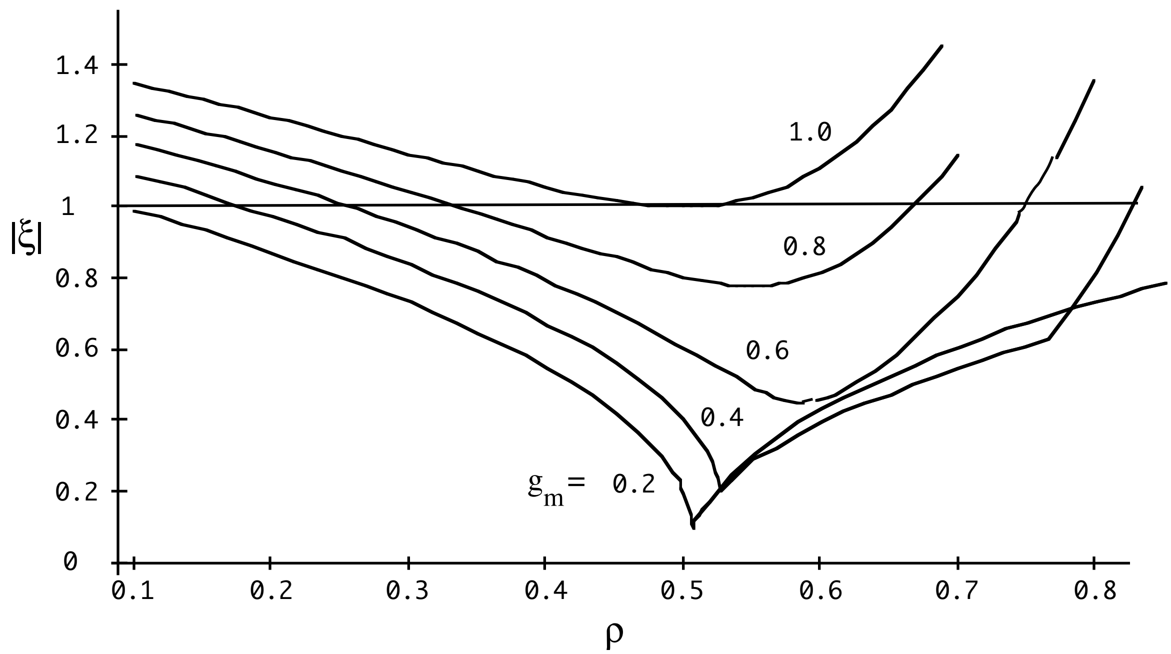


Figure 4 The dependence of the modulus of the amplification factor on reflection coefficient and convection number.

On the basis that the amplification factor will first reach the condition $|\xi| > 1$ when $\lambda\Delta x = \pi/2$, we can reduce our equation to:

$$\xi^3 - 2jB\xi^2 - (2\rho - 1)\xi + 2jB(2\rho - 1)^2 = 0 \qquad (20)$$

We therefore look at the largest modulus of the three roots and see how it is influenced by $\rho$ and $g_m$. The results are shown in figure 4. These indicate that for $\rho > 0.666$ then the algorithm is unstable for $g_m = 0.8$. This move towards instability can be seen in Figure 5.

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Figure 5 The effect of convection number on the accuracy and stability of TLM results

There have been suggestions that upwinding gives better results. We have certainly looked at a range of schemes and have been surprised to find that none of those that we have investigated are as good as the original Saleh method.

## 4. Some further observations

A current generator can be treated as a negative resistance in a network. Experience with microwaves would suggest that a negative resistance component does not of itself lead to oscillations so long as the overall impedance of the node remains positive over the full frequency spectrum. This might explain why instabilities have not been observed in TLM algorithms for linear Poisson equations. The case of non-linear Poisson equations have yet to be investigated.

There is still much work to be done on TLM algorithms for Laplace and Poisson problems and the interfacing with full-field Markov methods should be of particular interest. Lobry et al [6] have recently used TLM for boundary element method (BEM) solutions of open-boundary Poisson problems and I am very grateful to Xiang Gui for drawing my attention to this reference.

Xiang Gui has also drawn my attention to a paper which questions the validity of the telegraphers' equation for crystalline solids [7]. In conclusion, I believe that we have plenty of research material in this subject which will keep us busy for some time to come.

## 5. References


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