Stability of solution of the diffusion equation using various formulation of finite difference method

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Abstract

A probabilistic approach has been used to analyze the stability of the various finite difference formulations for propagation of signals on a lossy transmission line. If the sign of certain transition probabilities is negative then the algorithm is found to be unstable. We extend the concept to consider the effects of space and time discretisations on the signs of the coefficients in a probabilistic finite difference implementation of the Telegraphers' equation and draw parallels with the Transmission Line Matrix (TLM) technique.

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Introduction

The behavior of electrical signals on a lossy electrical transmission line can be used as a very reasonable approximation of the diffusion equation

$$\frac{\partial V}{\partial t} = K\nabla^2 V \tag{1}$$

with $K \equiv 1/(R_d C_d)$ where R_d and C_d are the distributed electrical parameters of resistance and capacitance. However, it has been shown [1] that an equation of this form cannot hold exactly, since being parabolic, it predicts an infinite velocity of propagation on the transmission-line analogue. This paradox can be avoided by adopting the Telegraphers' equation which contains an additional, but very significant term:

$$\frac{\partial V}{\partial t} + \tau\left[\frac{\partial^2 V}{\partial t^2}\right] = K\nabla^2 V \tag{2}$$

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In this case K is as it was defined in eqn (1), while $\tau = L_d/R_d$ can be interpreted as a relaxation time (L_d is distributed inductance). Eqn (2), a hyperbolic equation, predicts a finite value of propagation velocity. In the case of heat-flow the additional term in eqn (2) was discussed by Simons [1] on the basis of the Boltzmann transport equation.

In this paper we consider some finite difference treatments of eqns (1) and (2). We seek to interpret the resulting parameters in terms of transition probabilities. A probabilistic interpretation of eqns (1,2) has been presented by Kaplan [2] for the case of heat-flow. Depending on the exact formulation and the spatial and temporal discretisations there may be unconditional stability, conditional stability or unconditional instability and we investigate the extent to which these probabilistic analogies can be drawn. We will also draw parallels with the equations of the Transmission Line Matrix (TLM) method. This is a time domain numerical technique which avoids direct use of differential equations. The basic unit of the spatial mesh is the node which comprises an interconnection of transmission lines. The behavior of electrical voltage and current impulses on this network analogue is identical to that of signals on conventional transmission lines; they obey the same differential equations. Thus, time discretisation is automatically forced on the system. A network of transmission lines and resistors emulates the Telegraphers' equation and the implementation involves scattering parameters ρ (reflection) and τ (transmission) which have been shown to be equivalent to the transition probabilities in simple and correlated random walks [3]. The case where $\rho = \tau$ is equivalent to the treatment of Taitel [4], while the case where $\rho \neq \tau$ is equivalent to the correlated random walk of Goldstein [5]. We seek to broaden the investigation using a series of one-dimensional analyses, but in order to avoid any confusion in the use of symbols, we will retain τ for relaxation time and use the symbol Γ for the transmission coefficient of TLM.

Analysis

We start with a treatment that involves central difference discretisations of the derivatives on both sides of eqn (1). If time is expressed as $t = n\Delta t$ and spatial distance, $x = i\Delta x$, n and i being integers we have:

$$V(n+1,i) = V(n-1,i) + p_{x-}V(n,i+1) + p_{x+}V(n,i-1) + p_t V(n,i) \tag{3}$$

where $p_{x-} = p_{x+} = \dfrac{2K\Delta t}{\Delta x^2}$ and $p_t = -\dfrac{4K\Delta t}{\Delta x^2}$

The coefficients $p_{x-} = p_{x+}$ can be interpreted as the probability that a signal will move its spatial position by one point to either side of x during a single time interval. The coefficient p_t is interpreted as the probability that a signal will remain at position x during a single time interval. Since there is no way in which p_t as it appears in eqn (3) can be non-negative we believe that this is consistent with the well-known instability of such formulations. We have proved that there is no way in which solution of eqn (3) could be stable during calculation of V(n,i). We believe that this is a result of p_t being negative.

If we now consider a backward difference treatment where the time derivative is given by: [V(n,i) - V(n-1,i)]/Δt, then we obtain a slightly different expression for V(n+1,i):

$$V(n+1,i) = p_x\left[V(n,i+1) + V(n,i-1)\right] + p_t V(n,i) \tag{4}$$

In this case $p_x = \frac{K\Delta t}{\Delta x^2}$ (i.e. half the value of p_{x-} and p_{x+} in eqn (3)) and $p_t = \left[1 - \frac{2K\Delta t}{\Delta x^2}\right]$
The probability of remaining at point, (i) during one time interval is non-negative so long as
$\frac{K\Delta t}{\Delta x^2} \leq \frac{1}{2}$ (i.e. the stability limit of the backward finite difference formulation). We note that the sum of probabilities, $(p_{x+} + p_{x-} + p_t) = 1$. As can be seen from Fig. 1 calculated results obtained using eqn (4) coincide with those obtained from the Telegraphers equation and TLM central difference equation, provided that p_t is positive. If it is negative eqn (4) is non-stable.

We can also develop a forward difference treatment, where the time derivative is given by [V(n+1,i) - V(n,i)]/Δt, but in order for this to be meaningfully different the spatial double derivative must be based on the new time, i.e. (n+1)Δt. Thus:

$$\frac{\partial^2 V}{\partial x^2} = \frac{\left[V(n+1,i+1) + V(n+1,i-1) - 2V(n+1,i)\right]}{\Delta x^2}$$

The resulting implicit finite difference formulation is known to be unconditionally stable [6]. Our derivation yields:

$$V(n+1,i) = p_x\left[V(n+1,i+1) + V(n+1,i-1)\right] + p_t V(n,i) \tag{5}$$

where $p_x = \frac{\left[\frac{K\Delta t}{\Delta x^2}\right]}{\left[1 + \frac{2K\Delta t}{\Delta x^2}\right]}$ is positive and always less than unity

and $p_t = \frac{1}{\left[1 + \frac{2K\Delta t}{\Delta x^2}\right]}$ is positive and exists in the range $(0 < p_t \leq 1)$

Because of the implicit nature of the formulation these probabilities should be defined in a slightly different way from previously. In this case we are looking back from the time, (n+1)Δt and interpreting what has happened. Thus, p_{x-} is the probability that a signal has arrived at x from an adjacent position during the time-step. Similarly, p_{x+} is the probability that a signal has remained at x during the time-step. This definition implies that the sum $(p_{x+} + p_{x-} + p_t) = 1$, which it does. As p_x and p_t are positive eqn (5) is stable for arbitrary chosen Δt and Δx. However, signal values obtained have been found to be significantly lower (see Fig. 1) than those obtained from the Telegraphers' equation, the TLM formulation or from eqn (4).

We can now move to a consider the similar representations of the Telegraphers' equation. Eqn (2) can be analysed as a central difference approximation in all three derivatives as shown in eqn (6):

$$\frac{V(n+1,i) - V(n-1,i)}{2\Delta t} + \tau\left[\frac{V(n+1,i) - 2V(n,i) + V(n-1,i)}{\Delta t^2}\right] = K\left[\frac{V(n,i+1) - 2V(n,i) + V(n,i-1)}{\Delta x^2}\right] \tag{6}$$

This can be recast in a probabilistic form as:

$$V(n+1,i) = p_{2t}V(n-1,i) + p_t V(n,i) + p_x[V(n,i+1) + V(n,i-1)] \tag{7}$$

where the coefficients (can be interpreted as follows):

p_x is interpreted as the probability that signals will be transferred from nodes (i-1) and (i+1) to node (i) during Δt.

$$p_x = \frac{2K\left(\frac{\Delta t}{\Delta x}\right)^2}{\Delta t + 2\tau} \tag{8}$$

It is dimensionally similar to the equivalent transition probability for eqn (3): $2K\Delta t/\Delta x^2$.

p_t is the probability that a signal will remain at node, (i) during Δt

$$p_t = \frac{4}{\Delta t + 2\tau}\left(\tau - K\frac{\Delta t^2}{\Delta x^2}\right) \tag{9}$$

This is quite different from the equivalent probability for eqn (3): $-4K\Delta t/\Delta x^2$.

p_{2t} is the probability that a signal which is at node (i) will still be at node (i) two time-steps later (see Fig. 1) and is given by.

$$p_{2t} = \frac{\Delta t - 2\tau}{\Delta t + 2\tau} \tag{10}$$

We see in this formulation that $(p_{x+} + p_{x-} + p_t + p_{2t}) = 1$. We can now revert back to eqn (3) to observe that the coefficient in V(n-1,i), $p_{2t} = 1$, while $(p_{x+} + p_{x-} + p_t) = 0$. The same would apply in eqn (10) if $\tau = 0$, i.e. infinite speed of propagation. Nevertheless, we note that there are conditions of discretisation and relaxation time when these probabilities could be negative and these are considered later. A series of numerical experiments that have been undertaken in parallel with this work has demonstrated that eqn (7) is stable so long as $p_t > 0$. We find that the stability of the formulation does not depend on the sign of p_{2t}. This statement should be proved rigorously but we feel that it is beyond the scope of this paper.

The interpretation of the coefficients p_t and p_x appearing in eqns (4) and (7) can be verified numerically. Fig. 1 presents their changes with the variation of the distance between the neighboring node distance Δx. As the distance grows the probability p_t that a signal will remain at position x during a single time interval approaches one. However, the probability p_x that a signal has arrived at x from the neighborhood during the time-step decreases as Δx grows.

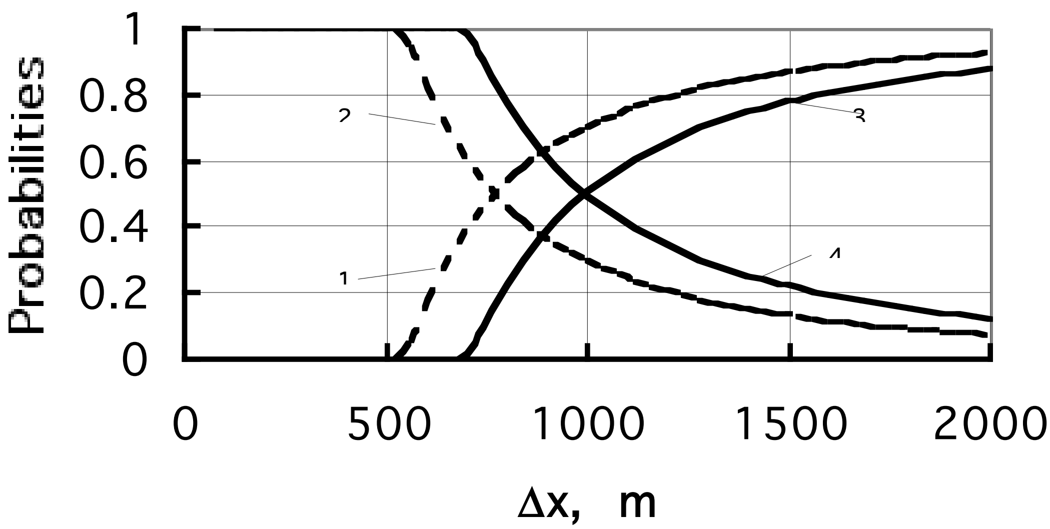

Fig.1. Probability coefficients vs the neighbouring node distance Δx. Curves 1 and 2 present p_t and $2p_x$ of eqn (4) respectively and curves 3 and 4 present $p_t + p_{2t}$ and $2p_x$ of eqn (7) respectively.

The Telegraphers' equation can be approximated by means of a central difference scheme for the two double derivatives with backward difference representation for the single time derivative. This yields an expression which is identical with eqn (7) but where the coefficients are

$$p_x = \frac{K\left(\dfrac{\Delta t}{\Delta x}\right)^2}{\Delta t + \tau} \qquad p_t = \frac{\Delta t + 2\tau - 2K\left(\dfrac{\Delta t}{\Delta x}\right)^2}{\Delta t + \tau} \qquad \text{and} \qquad p_{2t} = \frac{-\tau}{\Delta t + \tau} \tag{11}$$

Once again $(p_{x+} + p_{x-} + p_t + p_{2t}) = 1$, but we note that p_{2t} is unconditionally negative and we are forced to ask what this means, particularly as such formulations can be stable for

p_t positive. Some clue may be found in the one-dimensional formulation of the Telegraphers' equation using the Transmission Line Matrix (TLM) technique. The computational space is represented by a discretised network of transmission lines (impedance, Z) and series resistors (magnitude 2R). It can be shown that the impedance $Z = \Delta t/C = L/\Delta t$ (where C is the capacitance given by $C_d\Delta x$ and L is the inductance given by $L_d\Delta x$). Similarly, $R = R_d\Delta x$. C_d, L_d, and R_d are the distributed capacitance, inductance and resistance of the transmission line, respectively. The reflection coefficient of a lossy line is given by $\rho = R/(R + Z)$ and the transmission coefficient by $\Gamma = Z/(R + Z)$. The voltage at node i and time n+1, $\phi(n+1,i)$ is given in terms of local values and previous times as

$$\phi(n+1,i) = \Gamma\left[\phi(n,i+1) + \phi(n,i-1)\right] + \left(\rho^2 - \Gamma^2\right)\phi(n-1,i) \qquad (12)$$

This is a two-step Markov process [7] with a correlation coefficient $(\rho^2 - \Gamma^2)$ which, although it can range in value between +1 and -1, is unconditionally stable [8]. Since $(\rho + \Gamma) = 1$ we can rewrite eqn (12) as

$$\phi(n+1,i) = p'_x\left[\phi(n,i+1) + \phi(n,i-1)\right] + p'_{2t}\,\phi(n-1,i) \qquad (13)$$

where $p'_x = \Gamma$ and $p'_{2t} = (\rho - \Gamma)$, so that $(p'_{x+} + p'_{x-} + p'_{2t}) = \rho + \Gamma = 1$. Note that $p'_t = 0$ as there is no $\phi(n,i)$ in eqn (12).

It can be shown that $p'_x = \Gamma = \dfrac{K\left(\dfrac{\Delta t}{\Delta x}\right)^2}{\tau + \Delta t}$

It can also be shown that $p'_{2t} = (\rho - \Gamma) = \dfrac{\Delta t - \tau}{\Delta t + \tau}$

We can now return to the finite difference discretisations of the Telegraphers' equation and note the points of similarity with TLM. p'_x is identical with p_x in eqn (11). It is similar in form to eqn (8). p'_t is not defined for TLM. Now, if $\Delta t >> \tau$ then $p'_{2t} \rightarrow 1$, while if $\Delta t << \tau$ then $p'_{2t} \rightarrow -1$ and yet the algorithm remains unconditionally stable. In the central difference formulation p_{2t} differs from p'_{2t} by a factor 2 in τ (eqn (10)), but ranges between +1 and -1 depending on the relative magnitude of Δt and τ. In the backward difference formulation p_{2t} ranges between -1 and zero (eqn (11)). We therefore suggest that it is p_t and not p_{2t} which has the major influence on stability. It is always negative in eqn (3), conditionally positive in eqn (4) and unconditionally positive in eqn (5). In the next section we will therefore investigate the influence of discretisations on p_t.

The influence of discretisation parameters

Δx is the distance between nodes in our discretised model and Δt is the time interval between inspections of the contents of the nodes. So $\Delta x/\Delta t$ is a property of our model which has the units of velocity. We also have v, the velocity of the particles/charge/signal. It is given by $v = \lambda/\tau$, where λ is the distance travelled during the relaxation time, τ. We shall now use these to define the circumstances under which p_t may be either conditionally or unconditionally negative as functions of the discretisations and the parameters, K and τ.

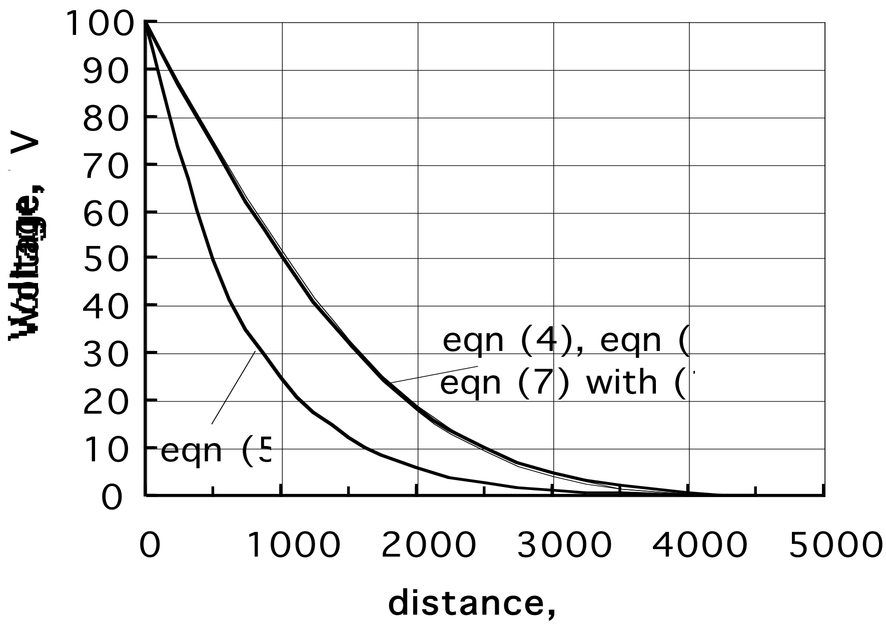

Fig. 2 Distribution of impulse height after 10μs along a 5000m transmission line with 20 nodes.

(i) is the result for eqn (5): forward difference formulation of the diffusion equation.

(ii) represents three coincident curves

- eqn (4) backward difference formulation of the diffusion equation
- eqn (7) central difference (in space and time) of the Telegraphers' equation
- eqn (11) Telegraphers' equation with combination of central and backward differences.

(calculations based on $L_d = 1.284\text{x}10^{-7}$ H/m, $C_d = 8.66\text{x}10^{-12}$ F/m and $R_d = 1\ \Omega$/m)

Using the relationship $K = v^2\tau$ the probability p_t can be presented in a more convenient form for this analysis:

Eqn (9) (central difference)

$$p_t = \frac{4\tau}{\Delta t + 2\tau}\left(1 - \frac{v^2}{\left(\frac{\Delta x}{\Delta t}\right)^2}\right) \tag{14}$$

Eqn (11) (backward difference) $$p_t = \frac{\Delta t + 2\tau\left(1 - \frac{v^2}{\left(\frac{\Delta x}{\Delta t}\right)^2}\right)}{\Delta t + \tau} \tag{15}$$

We will now consider the conditions for non-negative p_t in both of these circumstances.

Case 1. $\Delta x/\Delta t < v$ or $\lambda > \Delta x\left[\frac{\tau}{\Delta t}\right]$

In this condition p_t for the central difference is always negative. There is a conditionally positive region for backward difference scheme and this can be stated as:

$$\frac{\Delta x}{\Delta t} \geq \frac{v}{\sqrt{\frac{\Delta t}{2\tau} + 1}} \tag{16}$$

Case 2. $\Delta x/\Delta t = v$ or $\lambda = \Delta x\left[\frac{\tau}{\Delta t}\right]$

Here p_t is positive for eqn (15) and zero for eqn (14). We note that eqn (7) is now similar to the TLM scheme, although the finite difference scheme involves 2τ for p_x and p_{2t} while TLM involves τ in the equivalent expressions. Thus the central difference formulation will have non-negative p_{2t} so long as $\Delta t \geq 2\tau$, while for TLM the equivalent condition is $\Delta t \geq \tau$.

Case 3. $\Delta x/\Delta t > v$ or $\lambda < \Delta x\left[\frac{\tau}{\Delta t}\right]$

In this case p_t in eqns (14) and (15) are always positive. The behavior of parameter p_{2t} is the same as in Case 2. Figure 2 presents the distribution of the signal height along a 5000 meters long transmission line after elapsing 10μs. The line parameters are given in the figure caption. In order to compare the simulated results, there were chosen 20 nodes and the impulses of each of them after 10μs were calculated separately according to eqn (4), eqn (5), eqn (7) with coefficients (8)-(10), and eqn (7) with coefficients (11). In this way four curves have been obtained, which are shown in the figure. As can be seen, there occurs a coincidence between three of them except the curve obtained from eqn (5).

Because of the coincidence between three plots in figure 2 we decided to show that the difference plots. Figure 3 depicts three curves representing the absolute differences between the voltages at every node along the transmission line. There were chosen 20 nodes and the impulses of each of them after 10μm were calculated separately according to eqn (4), eqn (7) with coefficients (8)-(10), and eqn (7) with coefficients (11).

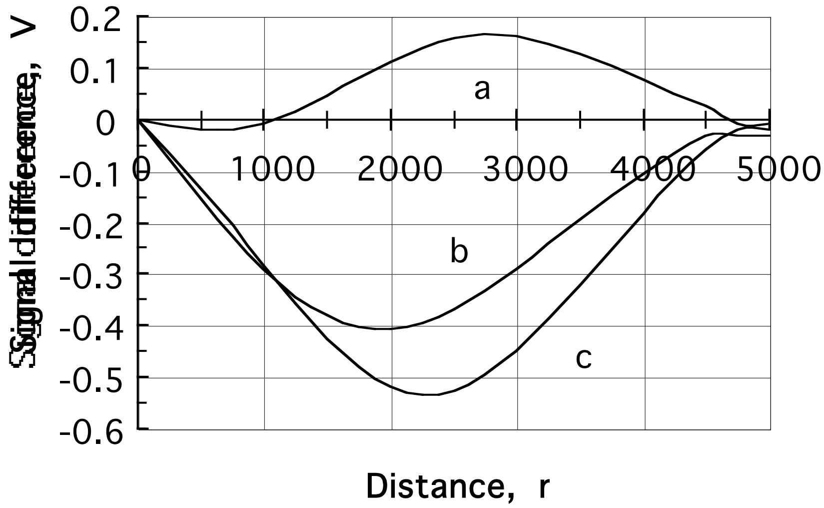

Fig. 3. Absolute differences between the voltages obtained from eqns (4) and (7) at every node along the transmission line. There were chosen 20 nodes and the differences of impulses of each of them after 10μm were calculated separately. (a) – difference between eqn (7) with coefficients (8)-(10) and eqn (7) with (11), (b) - difference between eqn (7) with coefficients (11) and eqn (4), (c) - difference between eqn (7) with coefficients (8)-(10) and eqn (4). The line parameters as in Fig. 1.

Conclusion

This work has shown that it is possible to use a transition probability approach to obtain information on the likely stability bounds in a particular central difference scheme for the Telegraphers' equation. The use of transition probabilities is not new in numerical solutions of such equations. They form the basis of the LISA method of Delsanto [3]. We have also drawn parallels with the Transmission Line Matrix method, which depending on the transition probabilities are analogues of simple or correlated random walk processes. These help us to suggest that p_{2t} in the finite difference formulations is a equivalent to the correlation coefficient of TLM and can assume negative values without giving rise to instability. The results of numerical analysis show that the following formulations of the diffusion and the Telegrapher's equation results the acceptable solutions: the backward difference, the central difference and Telegraphers' equation with combination of central and backward differences. However the forward difference formulation of the diffusion equation gives the different results. Future attention should therefore focus on the role of the p_t term in determining stability as a function of problem parameters and discretisations.

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