A combinatorial approach to dispersion patterns in lossless and near-lossless 2d TLM

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Abstract

This paper builds on some extraordinary work by William O'Connor which was presented at the TLM conference which was held at hotel Tina in Warsaw. Various observations have been made about the way in which the outer periphery as well as the inner bulk expand as iterations proceed following a single point excitation. O'Connor's analysis provides a firm theoretical basis, although it is not certain if it is the only explanation. It certainly indicates that the isolated lossless scattering of a pulse at a two-dimensional node has long-term consequences and that is supported by our observations here.

Note:
The O'Connor paper is available through www.dandadec.co.uk, but such is its relevance to this paper it has been necessary to cite large extracts from it. These are shown in blue font to distinguish them from the content of this paper

1 Introduction

Numerical dispersion is a problem with Transmission Line Matrix (TLM) routines as it is with other time-domain numerical methods. The response following a single shot excitation into a lossless two-dimensional system has a certain beauty and is quite unexpected for those not familiar with time-domain scattering processes (figure 1).

At the heart of the scalar 2-D TLM algorithm is the simple scattering shown in figures 1a and 1b.

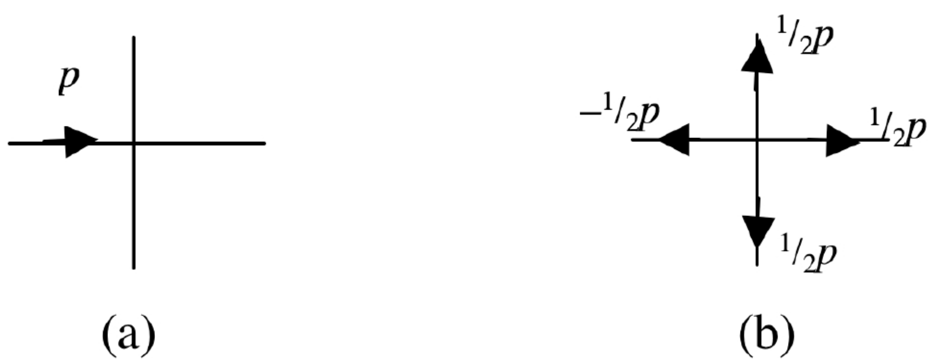

Figure 1 (a) single pulse of magnitude, p incident along the western arm of a two-dimensional shunt TLM node. (b) the subsequent scattering (one reflection and three transmissions) of this pulse

Figure 2 shows the result which is obtained sixty iterations after there is an initial excitation of magnitude p at the centre of the computational space.

How this "simple" scheme can represent wave phenomena has been extensively studied. The associated wave speed, dispersion relations, and so on, have been established using techniques based on circuit theory, linear analysis (matrix eigen-value problem formulation), and properties of periodic structures. The results of these approaches are usually presented in the frequency/wave domain.

The time-domain performance of the algorithm, meanwhile, remains somewhat unexplained. Some intriguing questions deserve more direct attention. For example:

- A question that must strike every newcomer to TLM is: If the pulses travel at $\Delta l/\Delta t$, and if the only thing that happens to them is the repeated application of Fig.1, where exactly, in the time domain, does a wave speed of $\sqrt{1/2}\ \Delta l/\Delta t$ come from? Can figure 1 alone "explain" a) the slow wave speed and b) the factor of precisely $\sqrt{1/2}$, particularly for wave propagation in the axial direction?
- A waveform in TLM is made up of a series of pulses, like figure 1a, each of which is continually "dispersing" (spreading out) further and further with every time step. This is evident from

repeated application of Fig.1. Yet somehow each pulse must also retain some “coherence”, as otherwise wave propagation would not be possible, not even at very long wavelengths, as all crests and troughs would gradually merge into each other. So, which is it? continuous dispersion? or retention of coherence? And if (as must be the case) the pulses somehow do *both*, what exactly is the mechanism in the time domain?

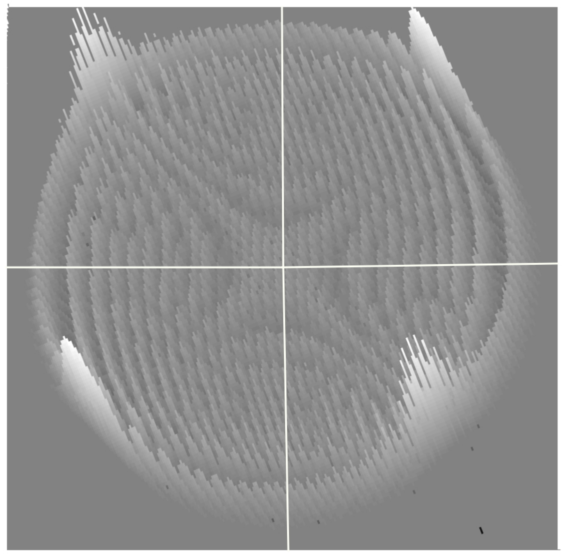

Figure 2 View of the apparent pressure profile after 60 iterations following an initial input at the centre of the two-dimensional mesh with p = 1/4 along each of four directions -x, y, x and -y

- TLM modellers often excite their model not with a single impulse but with a Gaussian distribution of pulses (a series of pulses whose envelope in space has the shape of a Gaussian curve). This cluster of pulses are then found to propagate without dispersion. The explanation of this coherence in the *frequency* domain is familiar: the problematical high frequency components in the “impulse” have been removed: the low frequency components then propagate without dispersion. But again, each component pulse in the Gaussian envelope is (as it were) “unaware” of its neighbours, “unaware” that it is part of a wider Gaussian envelope. Further each of these component pulses is “dispersing” indefinitely. So what is the time-domain method by which the overall distribution retains its coherence? In other words, how does each of the component pulses independently maintain the correct speed and shape to preserve the overall shape?

Somehow the scattering of figure 1 must tell the whole story: wave speed, dispersion, apparent impedance, capacitance and inductance, …and all without leaving the time domain.

Close consideration of figure 2 yields additional information. In figure 3 all non-zero nodes within a two-dimensional model space have been incremented by one. This has the effect of taking very small, but non-zero contributions and making them visible. So, sixty iterations after a single injection we see that there are scattered components at x = ±60, y = 0 and at x = 0, y = ±60. We see that the 'significant' circle that radiates from the source is circumscribed by square region with apices along the co-ordinate axes.

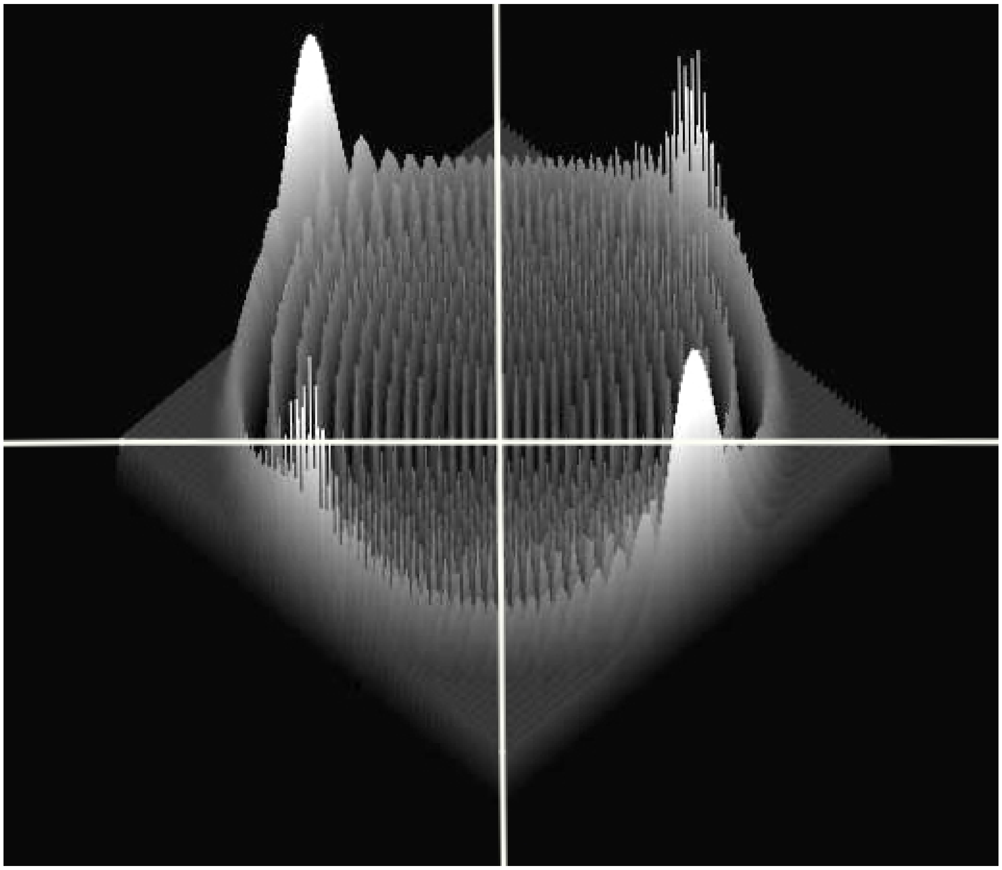

Figure 3 Pressure response as per figure 2 where a contribution of unit magnitude has been added to all non-zero nodes within the model space

If we consider the conditions that lead to figure 2 we find that even after a relatively small number of iterations the sum of all the pressures along the periphery soon approaches a value of +2 while the sum of the pressures at all nodes behind the periphery tends towards -1 so that at all times we have conservation of the initial unit injection. Figure 4 demonstrates this convergence during the first 13 iterations following injection. In the figure y1 represents the sum of pressures along the 'significant' periphery as seen in figure 2. There is a rising transient of the form

$$\text{sum of pressures} = 2\left[1 - e^{-rn}\right] \qquad (1)$$

where n is the number of iterations. This slope can be estimated as 0.69315 (it is interesting to note that $\log_e 2 = 0.693147$).

y2 and y3 in figure 4 represents the sum of all negative and positive pressures (nodal potentials) in the space behind the periphery and the sum (y2 + y3) has a transient of the form

$$\text{sum of pressures} = -1\left[1 - e^{-rn}\right] \tag{2}$$

Where the slope is identical to that which was observed along the periphery

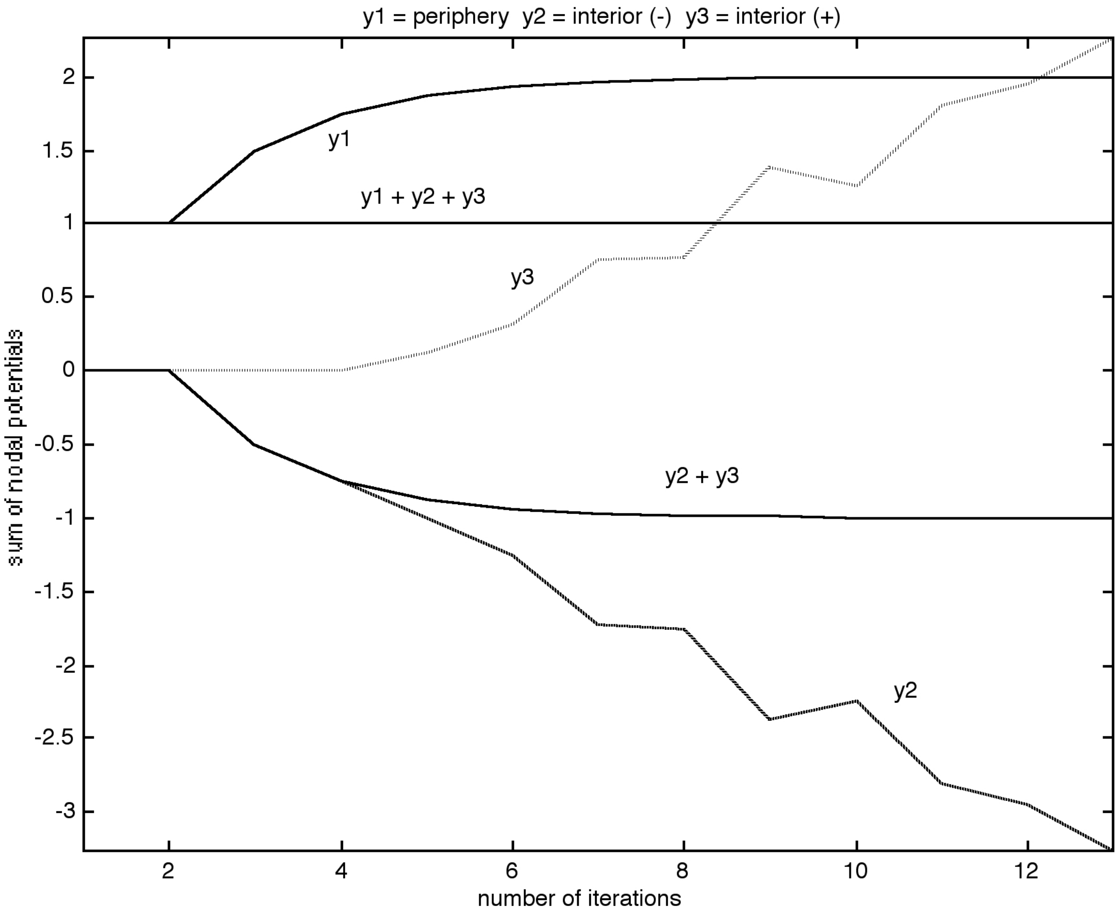

Figure 4 The variation of groupings of contributions as a function of iteration number following a unit initial injection:
y1 is the sum of all pressures along the 'significant' periphery of figure 2
y2 is the sum of all pressures with negative magnitudes behind the 'significant' periphery
y3 is the sum of all pressures with positive magnitudes behind the 'significant' periphery
y2+y3 is the net sum of pressures behind the periphery
y1+y2+y3 is the total sum of pressures along the periphery and behind

Everything that is observed here can be attributed to dispersion in a spatially discretised system. O'Connor [1] has given a microscopic-level electromagnetic interpretation for the scattering of pulses which at the macro-level yields figure 2.

The calculations which were used in figure 4 did not take account of the regions beyond the 'significant' periphery which are amplified in figure 3. We argue that these are indeed very small and will confirm this in the combinatorial approach which is described below.

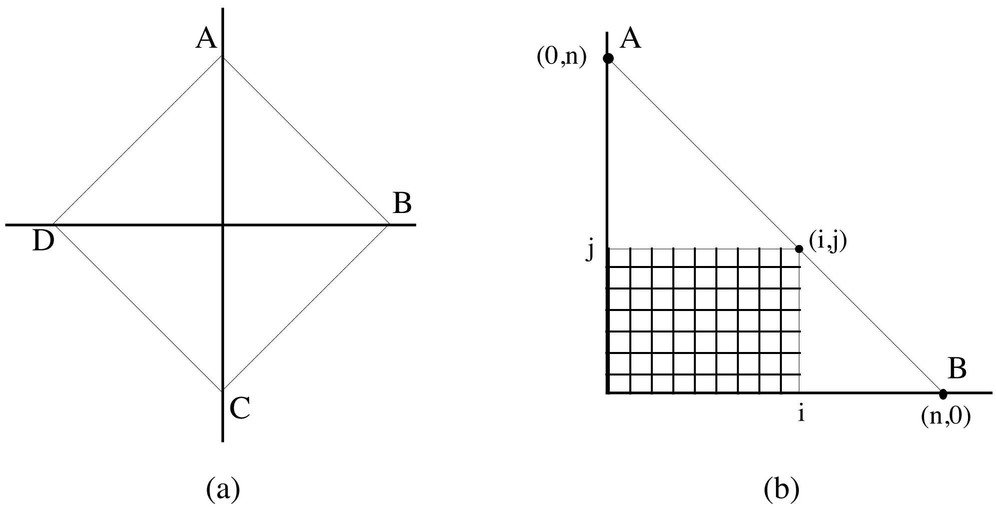

Figure 5 The outer extremities of a TLM simulation which has been run for n iterations: (a) the entire computational space, (b) detail of one quadrant showing the point (i,j) on the (A - B) peripheral segment.

2 A combinatorial approach to TLM scattering in two-dimensions

2.1 The sum of all components at the periphery of the computational space

We start with a two-dimensional space as shown in figure 5. Let us imagine that at an initial time we have an injection of 1/4 (unit) along each of the four transmission line directions leading to node (0,0), so that at time-step 1 we have at each neighbour vertex one reflection and three transmissions from the other lines. The resultant is $\rho/4 + 3\tau/4 = 1/4$ since $\rho = -1/2$ and $\tau = 1/2$ in lossless 2-D TLM models, but for what is discussed below we will not yet assign values to the scattering parameters.

These signals propagate through the network so that at time, n the points (i,j) that can be reached are all points such that $|i| + |j| \le n$. Consider the segment (A,B) in figure 5a. It can be seen that all contributions to the signal at the point marked *(i,j)* at time, *n* are within the rectangle which is shown in figure 5b and are accompanied by two possible moves, Right and Up and must contain *i* Right moves and *j* Up moves. Any point along the line AB contains n moves and the number of paths of length n is simply the number of ways of selecting i positions (positions of right moves) in a string of i + j characters (the j positions not selected are Up moves).

We have $\binom{n}{i}$ such moves. These paths do not contain any reflection and as the population of the first leg of each path was 1/4 the value of each path is then $\frac{1}{4}\tau^{n-1}$ (3)

The total value along the segment (A,B) is

$$\frac{1}{4}\tau^{n-1}\sum_{i=0}^{n}\binom{n}{i} = \frac{1}{4}\tau^{n-1}2^{n} = \frac{1}{2}(2\tau)^{n-1} \tag{4}$$

The total value along all four segments is then

$$V_n(\tau) = 2(2\tau)^{n-1} - 4(\frac{1}{4}\tau^{n-1}) \tag{5}$$

The second term on the right-hand side is because the extremities of the segments were counted twice.

We consider three cases

If $\tau < \frac{1}{2}$ then $\lim_{n\to\infty} V_n(\tau) \to 0$

which is what we should expect from such a physical process; the discontinuity between the border and the points out of reach at step, n disappears as n goes to infinity.

In the hypothetical situation where $\rho < -\frac{1}{2}$ $\tau > \frac{1}{2}$ then $\lim_{n\to\infty} V_n(\tau) \to \infty$

This means that if the signal value at the border increases to $+\infty$ then the interior of the signal area decreases to $-\infty$ since the signal values sum to 1.

Finally, when $\tau = \frac{1}{2}$ then $V_n(\tau) = 2 - \frac{1}{2^{n-1}}$

which is exactly what is observed in eqn (1). In this case we have $V_1(\tau) = 2$ and this converges rapidly to 2 as n $\rightarrow \infty$, whereas the sum of all the values in the interior is zero at n = 1 and decreases to -1 as the number of iterations increases.

2.2 Signal distribution along the periphery

We will now look at the (A,B) segment and consider the value of the signal at the point (i,n-i) in figure 6. This is given by

$$\binom{n}{i}\frac{1}{4}\tau^{n-i} \tag{6}$$

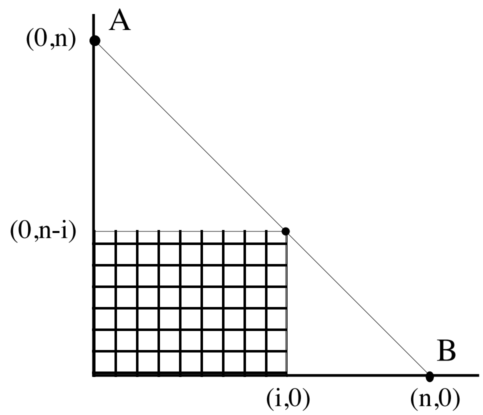

Figure 6 Designation of points along the AB periphery in the positive quadrant where the number of iterations n = i (in the x-direction) + j (in the y-direction)

Thus the values on (AB) are distributed with binomial coefficients, indeed, the distribution is identical to a binomial distribution with $\rho = 1/2$, when $\tau = 1/2$. For large values of n, then this is tends to a Gaussian distribution.

We can now consider a TLM-like process with $\tau = 1/2$ where successive scattering has been run for 2n iterations. The value at the point (n,n) is maximum along the (AB) segment and is equal to

$$\binom{2n}{n}\frac{1}{4}\left(\frac{1}{2}\right)^{2n-1} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n+1} \tag{7}$$

For n large Stirling's formula gives $\binom{2n}{n} \cong \frac{2^{2n}}{\sqrt{2n}}\frac{4}{5}$ (8)

Table 1 provides a confirmation of this for a wide range of n values

so that the value at (n,n) is approximately $\frac{1}{2\sqrt{2n}}\frac{4}{5} = \frac{2}{5\sqrt{2n}}$ (9)

and the average value of points along the segment is $\frac{1}{2(2n+1)}$ (10)

A comparison between the results obtained from (7) and the true values obtained from (8) are shown in table 2.

Table I

n	$\binom{2n}{n}$	$\frac{4}{5}\left(\frac{2^{2n}}{\sqrt{2n}}\right)$	relative error
4	70	72.4	0.034
8	12870	13107.2	0.018
16	601080390	607400100	0.011
20	137846528820	139078442305	0.009
32	1.8326×10^{18}	1.8446×10^{18}	0.007
128	5.768×10^{75}	5.7896×10^{75}	0.004

Table 2

n	time = 2n	$\frac{2}{5\sqrt{2n}}$	$\binom{2n}{n}\left(\frac{1}{2}\right)^{2n+1}$
2	4	0.200	0.1875
3	6	0.163	0.156
4	8	0.141	0.137
5	10	0.126	0.123
6	12	0.115	0.113
7	14	0.107	0.105
8	16	0.100	0.0982
9	18	0.094	0.093
10	20	0.089	0.088
11	22	0.085	0.084
12	24	0.082	0.081
13	26	0.078	0.077

The above expressions confirm that if $\tau = 1/2$ then the values decrease polynomially. If however $\tau \neq 1/2$ then this is not the case. We still have a peak at (n,n) but it decreases more rapidly than when $\tau = 1/2$. The maximum border value is approximated by

$$\frac{2}{5\sqrt{2n}}(2\tau)^{2n-1} \tag{11}$$

and the average value along the segment is

$$\frac{1}{2(2n+1)}(2\tau)^{2n-1} \tag{12}$$

so that when τ < 1/2 both the maximum and mean values decay exponentially. For example, if τ = 0.45 then after 60 iterations the value will be 2×10^{-3} of the value that would be obtained if τ were 0.5. When the simulation time is doubled (120 iterations) then relative magnitude is 3.6×10^{-6}; doubling the number of steps roughly squares the ratio. Even if τ is very close to 0.5 then this phenomenon is rapidly noticeable. For τ = 0.49 the ratios are as follows:

$$\frac{\text{value}(\tau = 0.49)}{\text{value}(\tau = 0.49)} = \frac{9}{100} \text{ (at n = 120) and } \frac{8}{100} \text{ (at n = 240)}$$

3 A combinatorial analysis of impulse velocity along the periphery

Whereas the pressure at a node in a two-dimensional TLM mesh is given by

$$P = \frac{{}^iV_{xp} + {}^iV_{xn} + {}^iV_{yp} + {}^iV_{yn}}{2\Delta x} \tag{13}$$

The pulse velocity in the x and y directions are given by

$$v_x = \frac{V_{xn} - V_{xp}}{Z\Delta x}$$
$$v_y = \frac{V_{yn} - V_{yp}}{Z\Delta x} \tag{14}$$

Z is the transmission-line impedance, Δx is the spatial discretisations. The subscripts *xp*, *xn*. *yp* and *yn* are intended to indicate direction of motion: *x* and *y* indicate motion along either the x- or the y-axis, *p* and *n* denote motion in the positive or negative directions.

Figure 7 shows the scattering pulses in the positive quadrant. We see that at node (2,0) we have a pulse of magnitude $\tau^2/4$ travelling in the positive x-direction which will arrive at (3,0) at the next iteration. This is designated as

$$\frac{\tau^2}{4}\begin{pmatrix}1\\0\end{pmatrix}$$

Similarly, the pulse at node (0,2) is of magnitude $\tau^2/4$ travelling in the positive y-direction which will arrive at (0,3) is designated as

$$\frac{\tau^2}{4}\begin{pmatrix}0\\1\end{pmatrix}$$

At the next iteration we will also have pulses incident at the other nodes along the new periphery: (2,1) and (1,2). The vector components of velocity that constitute these will be

$$\frac{\tau^2}{4}\begin{pmatrix}2\\1\end{pmatrix} \text{ and } \frac{\tau^2}{4}\begin{pmatrix}1\\2\end{pmatrix} \text{respectively}$$

A simple inspection shows that at the iteration after that the vector components of velocity, normalised to $\tau^3/4$ will be

$$\begin{pmatrix}0\\1\end{pmatrix} \quad \begin{pmatrix}1\\3\end{pmatrix} \quad \begin{pmatrix}3\\3\end{pmatrix} \quad \begin{pmatrix}3\\1\end{pmatrix} \quad \begin{pmatrix}1\\0\end{pmatrix}$$

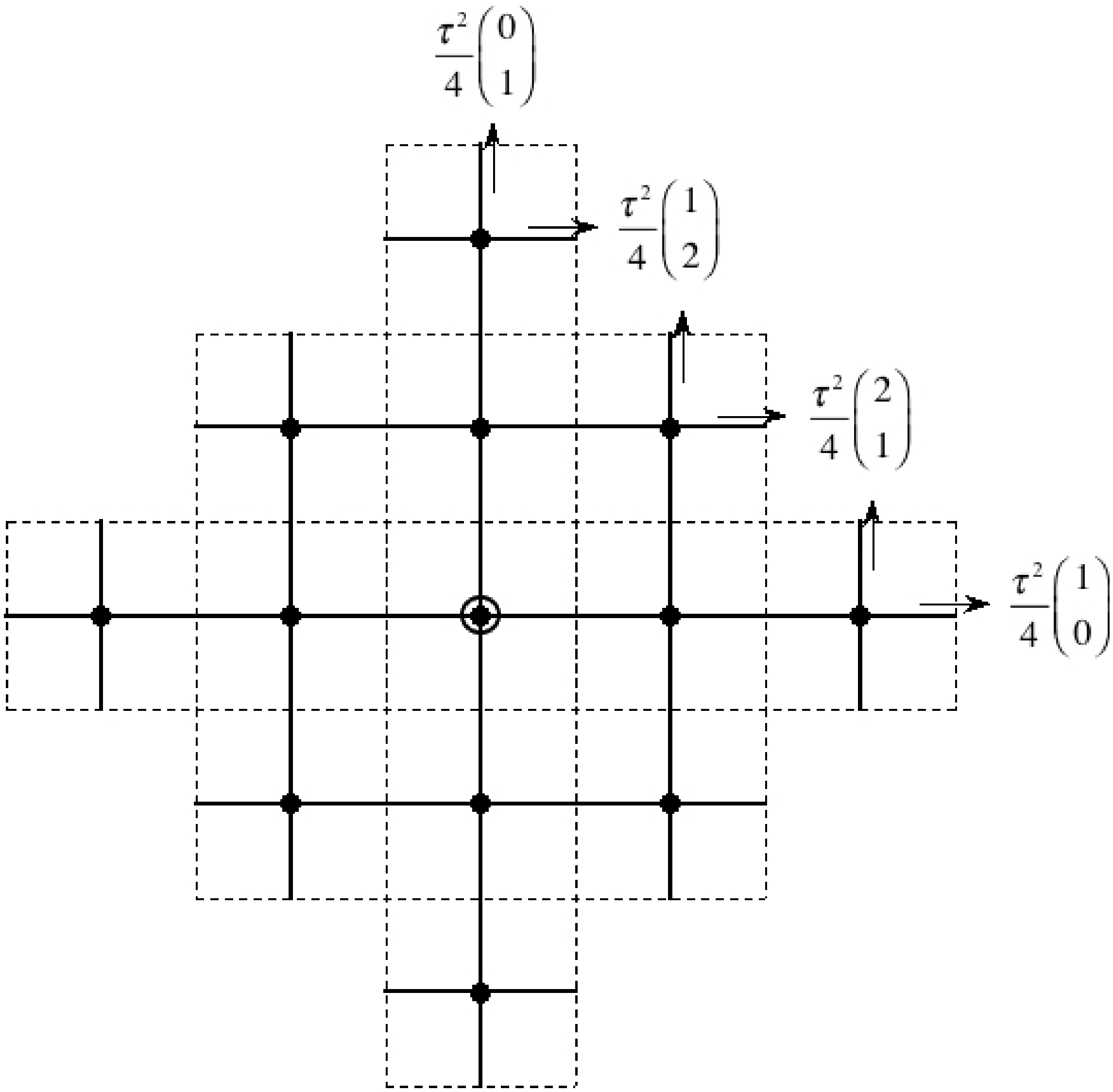

Figure 7 Pulses scattered from the periphery in the positive quadrant at n = 2

And for the iteration beyond that (normalised to $\tau^4/4$)

$$\begin{pmatrix}0\\1\end{pmatrix} \quad \begin{pmatrix}1\\4\end{pmatrix} \quad \begin{pmatrix}4\\6\end{pmatrix} \quad \begin{pmatrix}6\\4\end{pmatrix} \quad \begin{pmatrix}4\\1\end{pmatrix} \quad \begin{pmatrix}1\\0\end{pmatrix}$$

From which we can deduce that the velocity vector along the periphery for any iteration, n can be expressed by

$$\frac{\tau^{n-1}}{4}\begin{bmatrix}\begin{pmatrix}n-1\\i-1\end{pmatrix}\\\begin{pmatrix}n-1\\j-1\end{pmatrix}\end{bmatrix} \quad (15)$$

where n = (i + j) and where the components $\begin{pmatrix}n-1\\i-1\end{pmatrix}$ and $\begin{pmatrix}n-1\\j-1\end{pmatrix}$ are the binomial coefficients

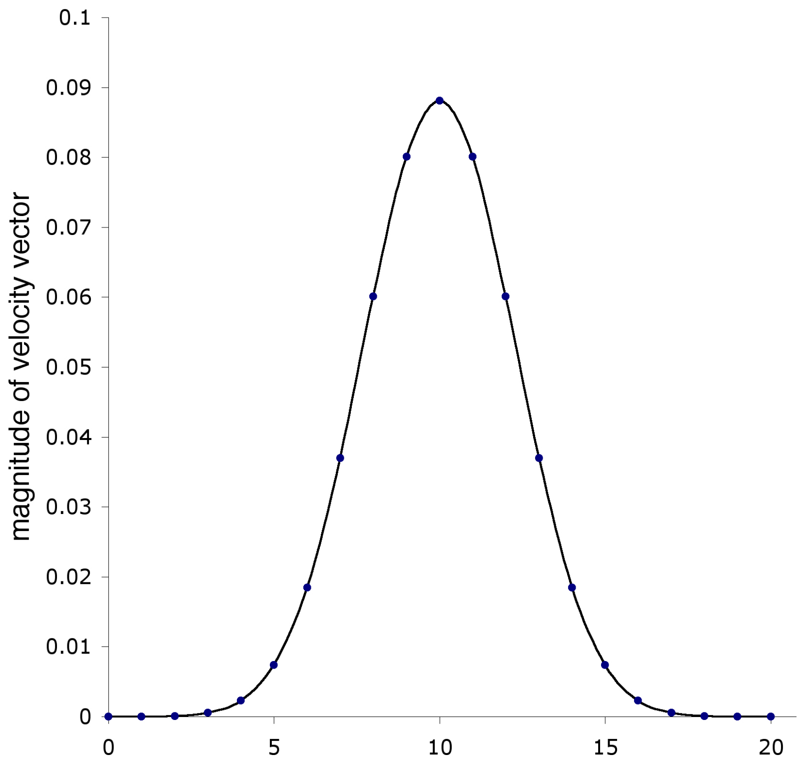

Figure 8 Magnitude of velocity (vertical axis) along periphery segment AB of figure 5 (horizontal axis) after 20 iterations

If we are only interested in the resultant direction of the velocity vector then this can be simplified to give

$$v_N = \begin{bmatrix}\frac{1}{j}\\\frac{1}{i}\end{bmatrix} \quad (16)$$

which can be seen in figure 8 and is in contrast to the direction of an equivalent wave-front in a non-dispersive system

$$v_N = \begin{bmatrix}i\\j\end{bmatrix} \quad (17)$$

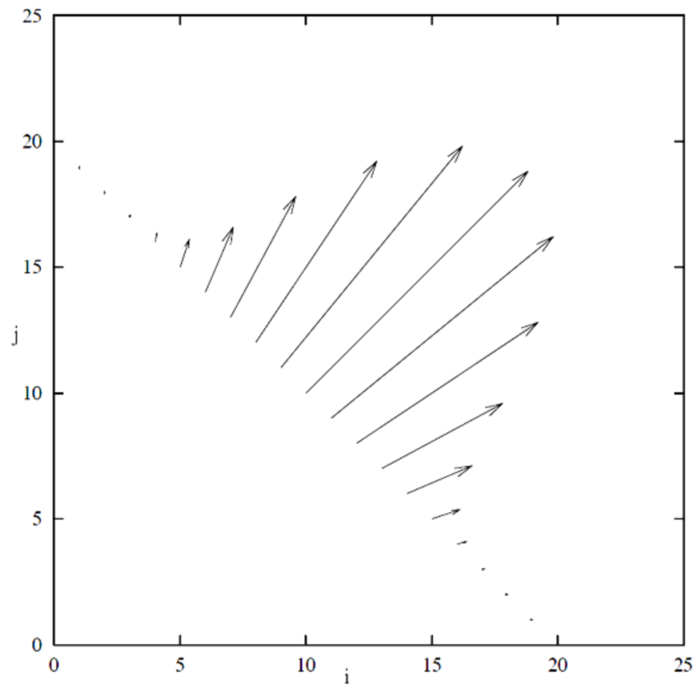

Figure 9 The vector orientation and relative magnitude along the AB segment of figure 5 after 20 iterations. The vertical and horizontal axes correspond to the i and j directions in figure 5(b)

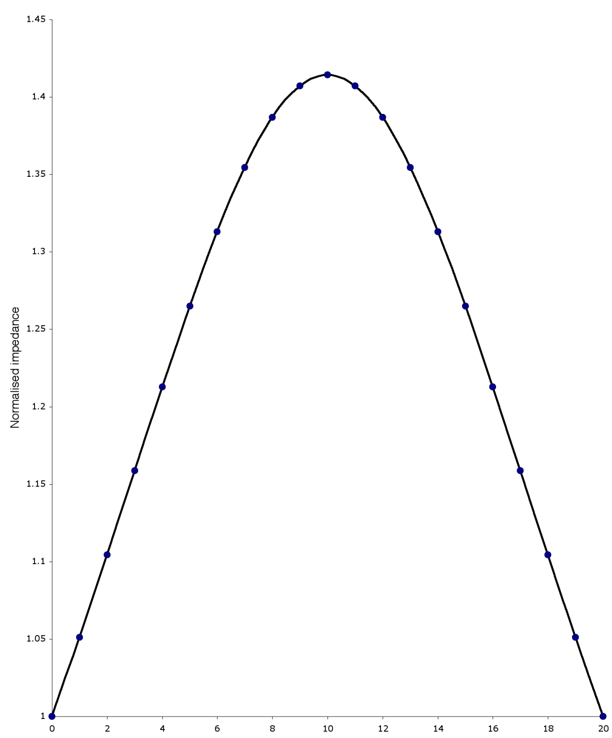

Figure 10 Wave impedance (vertical axis) along periphery segment AB of figure 5 (horizontal axis) after 20 iterations

4 A combinatorial analysis of wave impedance along the periphery

As the propagating wave reaches the 'far-field' its wave impedance should approach a fixed constant value. In the case of these simulations the far-field can be expected to be reached quite quickly. The impedance is given by

$$Z = \frac{P}{|v|} \tag{18}$$

This can be calculated analytically, but it can also be evaluated using the combinatorial approach. If we calculate the pressure along the AB periphery for n = 20 and insert the magnitude of the velocity vector as shown in figure 9 then we get a wave impedance, as shown in figure 10 which is in complete agreement with what would be calculated analytically

4 Conclusions

This and other work that we have presented elsewhere has demonstrated the fresh insights that can be obtained by the use of the combinatorial approach to the time history of lossless and near lossless TLM scattering following a single-shot initial excitation. Our analysis of the behaviour adds to the concern about one of the initial assertions about TLM, namely that it is a representation of Huygen's principle. We believe that this would be true only if there were a single wave-speed. We do indeed, see constructive interference of wavelets at the periphery as the wave-front due to the excitation develops. However, since the main front moves at a velocity $V_0/\sqrt{2}$, while an axis-dependent component travels at V_0 we must assume that this latter component is not available to provide perfect destructive interference behind the wave-front. This can be easily seen if one excites the centre of a mesh with a temporal Gaussian. As the front (with positive polarity) moves out one can see a wave with negative polarity following immediately behind. The original excitation point remains negative and seems reluctant to return to zero. The non-Huygens nature of lossless TLM has been commented on by Enders and Vannesete [2], but perhaps the best explanation, in terms of the requirement to maintain current and voltage pulse balance in the propagating wave has been given by O'Connor [1]

However a key feature of any conclusion must be a recognition that this approach yields results which are in agreement with [1], but the only point of commonality between the two methods is the scattering rule as shown in figure 1 which brings us back to O'Connor's original contention [1] that the entire physics and hence the entire space-time history is contained in the scattering rule.

References

1. W. O'Connor "The time-domain dynamics of 2-D TLM" in *Transmission Line Matrix (TLM) Modelling* Proceedings of a meeting on the properties, applications and new opportunities for the TLM numerical method (Hotel Tina, Warsaw 1- 2 October 2001) Edited D. de Cogan, School of Information Systems (UEA) 2002 (ISBN 1 898290 16 6)

2. P. Enders, C. Vanneste: "Huygens' principle in the transmission line matrix method (TLM). Local theory" International Journal of Numerical Modelling **16**(2) 2003 175 - 178